where $\rho_1'=\rho_1$, $\rho_2'=\rho_2\Lambda^2/(t+\Lambda)^2$. One can now go ahead and calculate M_{ij}' by using N/D method. An insight into the significance of the cutoff is obtained if one considers a one-channel calculation. In that case, before the cutoff is introduced,

$$T=N\bigg/\left(1-\frac{t}{\pi}\int\frac{N\rho}{t'(t'-t)}dt'\right) \tag{35}$$

and after the cutoff is introduced, we have

$$T=\bar{N}\bigg/\left(1-\frac{t}{\pi}\int\frac{\bar{N}\rho}{t'(t'-t)}dt'\right), \tag{36}$$

where $\bar{N}=N\Lambda^2/(t+\Lambda)^2$. Thus, introducing a cutoff is equivalent to modifying N to make it more convergent; also, $\bar{N}$ contains a "greater" amount of information; i.e., we are introducing additional interaction to make the integral convergent. In the two-channel calculation, $M_{12}=M_{21}=M_{12}'\Lambda/(t+\Lambda)$ and $M_{22}=M_{22}'\Lambda^2/(t+\Lambda)^2$. Again, we are introducing additional "interaction" in the form of a first-order pole at $t=-\Lambda$ for M_{12} and M_{21} and a second-order pole for M_{22}.

The calculations now proceed along the same lines as for the case of a sharp cutoff. The results from this are similar to the ones discussed in the main body of the paper.

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Self-Energy of the Electron*

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A perturbation theory is developed within the usual formalism of quantum electrodynamics which yields a finite unrenormalized electron Green's function and a finite value for the electron's electromagnetic self-mass in each order. This is subject only to the qualification in this paper, that the vacuum polarization is also obtained without divergences. Furthermore, the bare mass of the electron must vanish; the electron mass must be totally dynamical in origin.

I. INTRODUCTION

The empirical success of the renormalized perturbation solution of quantum electrodynamics has produced the hope that relativistic field theory can provide an adequate description of the physics of elementary particles. On the other hand, the infinities which are present in the perturbation expression for the unrenormalized quantities have made one cautious about taking the theory too seriously.

In this work we will show that these infinities are not intrinsic to the theory but are due to the inadequacy of the usual perturbation method. We will attempt to develop an alternate perturbation approach to quantum electrodynamics which yields finite results for the basic unrenormalized Green's functions. In addition, in the weak-coupling limit, we will give explicit expressions for these functions in the region far off the mass shell where ordinary perturbation theory fails.

This method will work only for a spin-$\frac{1}{2}$ fermion field coupled with a conserved current to a neutral vector field. Hence the results of this work will not be applicable to a general relativistic field theory.

In quantum electrodynamics there are only three divergences (the minimum) in the ordinary perturbation treatment and they are all "weak" in the sense of being only logarithmically dependent on cutoffs. They are summarized by the constants δm, $Z_1(=Z_2)$, Z_3. The divergence of the self-mass δm is just the analog of the classical electromagnetic mass divergence. The divergence of the wave-function renormalization constant Z_2 represents an incompatibility of the perturbation treatment of the interaction with the canonical commutation rule for the electron field. The divergence of the charge renormalization Z_3 represents a similar

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incompatibility for the electromagnetic field. The divergence of the vertex renormalization Z_1 is associated with the fundamental electron-field–electromagnetic-field interaction. As a consequence of gauge invariance $Z_1 = Z_2$ (Ward's identity).

The wave-function constant Z_2 has no physical significance, since because of charge conservation the electron field is not linearly coupled to any source. In contrast, Z_3 has an immediate physical significance, since the electromagnetic coupling to charges is linear in the field. Thus, the only physically meaningful divergences which arise in the usual perturbation treatment of quantum electrodynamics are Z_3 and δm.

Furthermore, the question of Z_3 really takes one outside of the scope of a closed physical theory. The electron couples to all charged systems by means of vacuum polarization. Thus, when we make an assumption about the high-energy behavior of the photon Green's function, we must keep in mind the fact that its form is influenced by all interactions. In this paper we shall consider only the mass question, which can be treated to some extent in the closed theory without the necessity of considering other systems.

The renormalizations are closely related to the asymptotic behavior of the electron and photon Green's functions far off the "mass shell." If the constants are *finite*, then these functions must have in the asymptotic region the same form as their uncoupled analogs, and hence are independent of the coupling constant. But it is precisely from this domain that the divergent contributions to the renormalization constants arise. Hence, it has long been supposed that the theory as formulated in the ordinary way is not consistent: that these free asymptotic forms for the Green's function are incompatible with the interaction. We shall first show that the physically uninteresting renormalization constant, $Z_2(=Z_1)$ can be made finite even in a perturbation treatment of the interaction, without the introduction of any divergent renormalizations, provided only that Z_3 is finite. This can be done if one makes a suitable choice of electromagnetic gauge. This means that we shall be able to write a *linear* integral equation for the vertex function Γ_μ, with a kernel whose "singular part" (the part of the kernel which gives divergences in perturbation theory) is expressed as a given power series. It is further an equation which does not contain divergences. In this case the only perturbation divergences are δm and Z_3, which are confined to the electron Green's function S and photon Green's function D. We shall then show that if the Schwinger-Dyson equations for the exact S in terms of S, Γ_μ, D is expanded as a series in S without the expansion of S or D, then the resulting equation for S will have finite nonperturbative solutions with no self-mass divergence provided only that the "mechanical" or bare electron mass vanishes and providing that Z_3 is finite. In a subsequent paper we shall show that we can also develop a finite perturbation theory for D within electrodynamics using the linear vertex equation that we shall obtain from our equation for S by using gauge invariance. The equations we shall study will provide us with the explicit forms for S and D far off the mass shell, expressed in terms of *constants* given as power series in the "bare" (unrenormalized) coupling constant which are finite term by term. The solutions we obtain will be valid for all values of the coupling constant provided that it lies within the assumed finite radius of convergence of the power series. In a preliminary account of this work[1] we stressed what was actually only a first-order approximation. In this paper we shall describe the general method which can be used to systematically compute all functions.

II. CHOICE OF GAUGE

In this section we will show that if Z_3 is finite, then a gauge can be chosen so that Z_1 has a finite expansion in a power series in the bare coupling constant $\alpha_0 = e_0^2/4\pi$. In the following sections, by employing this gauge in our calculation, we shall show that the electron self-mass is finite if and only if the "mechanical" electron mass vanishes. If Z_3 is finite, then the exact photon Green's function $D(k^2)$ has the property that

$$k^2 D(k^2) \to 1 \qquad (2.1)$$

as $k^2 \to \infty$. In all of the subsequent work, "asymptotic" will always mean for large space-like momenta. In this case, all integrations may be performed with a Euclidean metric. Since the dominant contribution to all radiative corrections comes when all Green's functions are far off the mass shell, we can compute the "divergent part" of the vertex function by using $D(k^2) = 1/k^2$, that is, in effect, by neglecting the photon self-energy. Of course, we must then establish that the theory does indeed provide for a finite Z_3. This will be established in a subsequent paper. In this case, it follows from earlier work[2] that in an arbitrary gauge of the form

$$D_{\alpha\beta} = [g_{\alpha\beta} - (k_\alpha k_\beta / k^2) G] D(k^2) \qquad (2.2)$$

if we include a cutoff defined by the replacement of D by

$$1/k^2 - 1/(k^2 + \lambda^2),$$

then Z_2 has the form

$$Z_2 = (\lambda^2/m^2)^{f(\alpha_0, G)} B(\alpha_0, G). \qquad (2.3)$$

f and B are functions only of the coupling constant α_0 and G. It has been shown[3] that if the gauge is changed, Z_2 changes in a simple and explicitly known way. If we let

$$\delta D_{\alpha\beta} = -\frac{k_\alpha k_\beta}{k^2} \gamma \left(\frac{1}{k^2 + \mu^2} - \frac{1}{k^2 + \lambda^2} \right), \qquad (2.4)$$

[1] K. Johnson, M. Baker, and R. Willey, Phys. Rev. Letters **11**, 518 (1963).

[2] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

[3] K. Johnson and B. Zumino, Phys. Rev. Letters **3**, 351 (1959).

then in this new gauge Z_2' is expressed in terms of Z_2 by the relation

$$Z_2'=Z_2(\lambda^2/\mu^2)^{(\alpha_0/4\pi)\gamma}. \quad (2.5)$$

μ is chosen to avoid infrared divergences, which are present also in B only because of the conventional way of defining Z_2. It is therefore obvious that a gauge change can be made so that Z_2' is independent of λ, and hence finite as $\lambda \to \infty$. That is because

$$Z_2'=B(\alpha_0,G)(\lambda^2/m^2)^{f(\alpha_0,G)}(\lambda^2/\mu^2)^{(\alpha_0/4\pi)\gamma}. \quad (2.6)$$

Thus, the transformation (2.4) with $-\alpha_0\gamma/4\pi=f$ brings us to a gauge where Z_2 is finite. It is a property of the "Landau"[4] gauge ($G=1$) that if f is computed it begins only in the fourth order; that is, in the second-order perturbation theory, in this special gauge, Z_2 is finite. However, if the corresponding calculations are made in the fourth order, logarithmic divergences are encountered. This means that the finite gauge in the fourth order has the form

$$D_{\alpha\beta}=[g_{\alpha\beta}-(k_\alpha k_\beta/k^2)G]D, \quad (2.7)$$

with $G=1+\alpha_0C_1$. In general,

$$G=1+\alpha_0C_1+\alpha_0^2C_2+\cdots, \quad (2.8)$$

where the series has the property of having finite numerical constants independent of any dimensional parameter. Now, if we make Z_2 finite by an appropriate choice of gauge, we also make Z_1 finite, if we maintain Ward's identity. Hence this choice of gauge permits a perturbation expansion for the vertex function which is finite. By making Z_2 finite, however, we do not necessarily make the S function finite in perturbation theory, because of the self-mass divergence. We shall turn to this problem in the next section.

III. ELECTRON SELF-ENERGY

We have shown in the previous section that if we neglect vacuum polarization, the perturbation expansion of the vertex function Γ_μ is finite in an appropriately chosen gauge. In this section we will see how to include the electron self-energy in a manner which maintains the finiteness of S and Γ_μ.

We begin with the Schwinger-Dyson equation[5] for $S(p)$.

$$\frac{1}{S(p)}=\gamma p+m_0+ie_0^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha S(p')\Gamma^\beta(p',p). \quad (3.1)$$

Now suppose there exist finite solutions to (3.1) which

FIG. 1. Equation for electron Green's function. The lines represent the exact S and D functions.

yield S and D functions having the same asymptotic form as the free Green's functions. Then in the appropriate gauge the expansion of Γ_μ in terms of the exact S and D must also be finite. Thus in Eq. (3.1) we can make such an expansion of Γ_μ without introducing any new infinities. Equation (3.1) then becomes

$$\frac{1}{S(p)}=\gamma p+m_0+ie_0^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha S(p')\gamma^\beta$$
$$-(ie_0^2)^2\int\frac{(dp')(dp'')}{(2\pi)^8}D_{\alpha\beta}(p-p')D_{\mu\nu}(p-p'')$$
$$\times\gamma^\alpha S(p')\gamma^\mu S(p'+p''-p)\gamma^\beta S(p'')\gamma^\nu+\cdots, \quad (3.2)$$

or in graphical form as expressed in Fig. 1. In the graphical expression we omit all graphs corresponding to expansions of S or D, since these are taken as exact in (3.2). Of course, we must remember that the gauge will also be defined in terms of a power series in α_0, so that in (3.2) we must also expand the gauge constant G. Since the gauge is defined to be that which makes Z_2 finite, and thus yield an S with the asymptotic form $1/\gamma p$ which is independent of α_0, the nth order of a gauge term in D which appears in the mth order of the formal expression (3.2) will then contribute to the $(m+n)$th order of the equation which we propose for S. For example, the true sixth-order terms in (3.2) are indicated graphically in Fig. 2.

We can make a corresponding expansion of the Schwinger-Dyson (S.D.) equations for the electron Green's function $S(A)$ in the presence of an external

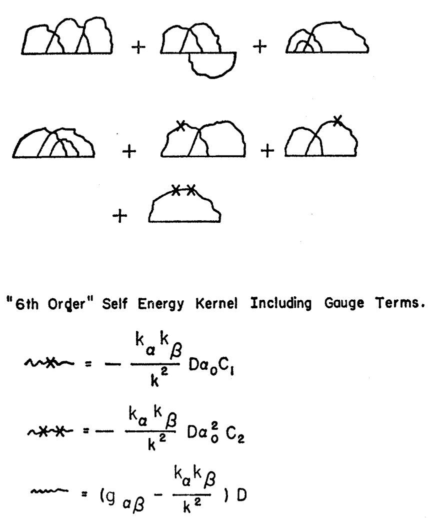

FIG. 2. "Sixth-order" self-energy kernel including gauge terms.

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[4] L. D. Landau, A. Abrikosov, and I. Halatnikov, Nuovo Cimento Suppl. **3**, 80 (1956).

[5] F. J. Dyson, Phys. Rev. **75**, 1736 (1949); J. Schwinger, Proc. Natl. Acad. Sci. U. S. **31**, 455 (1951).

field A_μ. If this equation is written in coordinate space and truncated at any order, the resulting truncated equation for $S(x,y;\,A)$ is invariant under the gauge transformation,

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda\,,\qquad S \rightarrow e^{ie_0[\lambda(x)-\lambda(y)]}S\,. \tag{3.3}$$

Now $S(x,y;A)$ determines the vertex function in coordinate space according to the definition,

$$\Gamma_\mu(xy;\,\xi) = -[\delta/\delta e_0 A_\mu(\xi)]S^{-1}(xy;\,A)\,. \tag{3.4}$$

From (3.4) and from our expansion for $S(x,y;A)$ corresponding to (3.2), we obtain a corresponding expansion for Γ_μ,

$$\begin{aligned}\Gamma_\mu(p+k,\,p) &= \gamma_\mu - ie_0^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha S(p'+k)\Gamma_\mu(p'+k,\,p')S(p')\gamma^\beta\\ &\quad+(ie_0^2)^2\int\frac{(dp')(dp'')}{(2\pi)^8}D_{\alpha\beta}(p-p')D_{\lambda\sigma}(p-p'')\gamma^\lambda[S(p''+k)\Gamma_\mu(p''+k,\,p'')S(p'')\gamma^\alpha S(p'+p''-p)\gamma^\sigma S(p')\\ &\quad+S(p''+k)\gamma^\alpha S(p'+p''-p+k)\Gamma_\mu(p'+p''-p+k,\,p'+p''-p)S(p'+p''-p)\gamma^\sigma S(p')\\ &\quad+S(p''+k)\gamma^\alpha S(p'+p''-p+k)\gamma^\sigma S(p'+k)\Gamma_\mu(p'+k,p')S(p')]\gamma^\beta+\cdots,\end{aligned} \tag{3.5}$$

or in graphical form as expressed in Fig. 3. Again the same remarks about the gauge expansion are relevant here. Equation (3.5) is a linear integral equation for Γ_μ whose kernel is a power series in the exact Green's function $S(p)$. If we truncate the expansions (3.2) and (3.5) at the same finite order, the solutions of the resulting approximate equations for S and Γ_μ satisfy Ward's identity exactly as a consequence of (3.3) and (3.4). Thus the approximate values for Z_2 and Z_1 obtained from these solutions for S and Γ_μ are equal. Hence if we choose the gauge in the truncated (3.5) so that Z_1 is finite, then in the same gauge Z_2 as obtained from the corresponding (3.2) will also be finite. In the following we will see under what conditions not only Z_2 but also the complete $S(p)$ as obtained from (3.2) is finite. First let us complete the above discussion by writing the equation for the D function. $D_{\alpha\beta}$ has the form

$$D_{\alpha\beta} = [g_{\alpha\beta} - (k_\alpha k_\beta/k^2)G]D\,; \tag{3.6}$$

we can write

$$1/D = k^2(1+\rho)\,, \tag{3.7}$$

where

$$\begin{aligned}(k^2g^{\mu\nu}-k^\mu k^\nu)\rho &= -ie_0^2\int\frac{(dp)}{(2\pi)^4}\,\mathrm{Tr}\gamma^\mu\\ &\quad\times\bigg[S(p+k/2)\Gamma^\nu(p+k/2,\,p-k/2)S(p-k/2)\\ &\quad+\frac{\partial}{\partial p_\nu}S(p)+\frac{1}{24}\left(k\frac{\partial}{\partial p}\right)^2\frac{\partial}{\partial p_\nu}S(p)\bigg].\end{aligned} \tag{3.8}$$

The additional terms in (3.8) are the contributions of the path integral from y to x used to make $S(x,y;\,A)$ gauge invariant before using it to generate an expression for the current.[6] The tensor structure of the left-hand side of (3.8), which expresses current conservation, is a consequence of the fact that the S and Γ_μ, which appear on the right-hand side of (3.8), are related by Ward's identity. The property of ρ necessary to ensure that $D\rightarrow 1/k^2$ as $k^2\rightarrow\infty$ is $\rho\rightarrow 0$. This property hinges upon the behavior of Γ_μ, which will be discussed in our next paper.

Let us now analyze the expression (3.2) for $S(p)$. We wish to find if it is possible that a solution to (3.2) exists which has the asymptotic property

$$1/S(p)\rightarrow \gamma p + m_0 \tag{3.9}$$

as $p^2\rightarrow\infty$. The general form for $1/S(p)$ is

$$1/S(p) = \gamma p(1+B(p^2)) + m_0 + A(p^2)\,. \tag{3.10}$$

Condition (3.9) is then equivalent to the conditions $B(p^2)\rightarrow 0$, $A(p^2)\rightarrow 0$. Thus, as $p^2\rightarrow\infty$, we must have

$$S(p)\rightarrow\frac{1}{\gamma p}[1-B(p^2)]+\frac{m_0+A(p^2)}{p^2}. \tag{3.11}$$

Perturbation theory yields in general logarithmically divergent expressions for $B(p^2)$ and $A(p^2)$. These

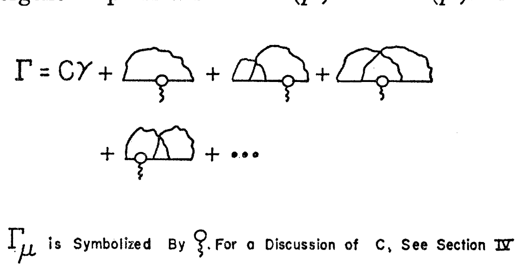

FIG. 3. Equation for vertex function.

[6] J. Valatin, Proc. Roy. Soc. (London) **A222**, 93, 228 (1954); J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).

divergences in the functions B and A can be isolated in terms of two divergent constants A' and B' which are the values of the functions $A(p^2)$ and $B(p^2)$ on the mass shell. A' and B' are gauge-dependent, while the self-mass $\delta m=-mB'+A'$ is independent of the gauge constant G. One might then hope to find a gauge in which the self-mass divergence is completely contained in A'. In such a gauge B' is finite and hence also the function $B(p^2)$. Of course this gauge is just the gauge which makes Z_1 finite in (3.5) as mentioned previously. Now the only source of the Dirac matrix **1** in our equations is the m_0 term in (3.2), since the coupling always introduces an even number of γ_μ matrices. Hence, it is clear that the self-mass divergence must be proportional to m_0. Indeed, we see that the most divergent contribution to A arises from inserting $1/\gamma p+m_0/p^2$ for S in the right side of (3.2) and retaining the terms linear in m_0. This suggests that if $m_0=0$, then perhaps the term A can "generate itself." That is, the resulting homogeneous equation for A may have a nontrivial solution. In this case perhaps a completely finite solution to (3.2) is possible. We will show that this is indeed the case if we in addition continue to assume that $D(k^2)\to 1/k^2$ as $k^2\to\infty$.

We being by analyzing the first approximation to (3.2).

$$\frac{1}{S(p)}=\gamma p+m_0+ie_0^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha S(p')\gamma^\beta. \quad (3.12)$$

Under the assumption (to be verified) that the asymptotic form of $1/S(p)$ is completely characterized by the contributions from large p' to the integral in (3.12), we get from (3.10), (3.11), and (3.12),

$$\gamma p B(p^2)+A(p^2)=ie_0^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha\left[\frac{1}{\gamma p'}[1+B(p'^2)]+\frac{m_0+A(p'^2)}{p'^2}\right]\gamma^\beta$$

which is valid as $p^2\to\infty$.

The above equation then separates into

$$B(p^2)\gamma p=ie_0^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha\frac{1}{\gamma p'}\times\gamma^\beta[1+B(p'^2)], \quad (3.13)$$

$$A(p^2)=ie_0^2\int\frac{(dp')}{(2\pi)^4}\gamma^\alpha\gamma^\beta D_{\alpha\beta}(p-p')\left(\frac{m_0+A(p'^2)}{p'^2}\right), \quad (3.14)$$

as $p^2\to\infty$.

Equations (3.13) and (3.14) then serve to determine the leading contributions to $A(p^2)$ and $B(p^2)$ for large p^2. If we put $D(k^2)=1/k^2$, (3.13) yields a logarithmically divergent B, unless we choose $G=1$. In this gauge (the Landau gauge), the contribution of the asymptotic form of D, i.e., $1/k^2$, to B according to (3.13) is zero, and hence the asymptotic form of B depends upon the leading deviations of $D(k^2)$ from $1/k^2$ at high k^2. For example, if $D-1/k^2\to(1/k^2)(1/k^2)^\epsilon$ then one can show from (3.13) that $B(p^2)\to(1/p^2)^\epsilon$ for small ϵ. In any case in the Landau gauge the infinities in (3.9) appear only in the equation for $A(p^2)$. However, if $m_0=0$, then (3.14) has perfectly finite solutions. For in that case it becomes

$$A(p^2)=-3ie_0^2\int\frac{(dp')}{(2\pi)^4}D(p-p')\frac{A(p'^2)}{p'^2}. \quad (3.15)$$

Equation (3.15) is solved in Appendix A. For small α_0 the solution is

$$A(p^2)=A_0(1/p^2)^{3\alpha_0/4\pi}, \quad (3.16)$$

where A_0 is an undetermined constant. In obtaining (3.16) only the asymptotic form of D, $(1/k^2)$, was used. If $D-1/k^2\to(1/k^2)(1/k^2)^\epsilon$, then this would produce corrections to A of the form $(1/p^2)^{3\alpha_0/4\pi+\epsilon}$. Now we must add to Eq. (3.2) the condition that the electron has a finite rest mass m; $1/S=0$ when $\gamma p=-m$. Since there is no input mass or scale in the theory we are free to choose the scale of energy to be the physical mass m of the electron. Then we require that $1/S=0$ when $\gamma p=-1$, and that $1/S$ has no other zeros. In order to impose this condition, it is necessary to investigate Eq. (3.2) in the nonasymptotic region. That is, if we imagine integrating (3.2) down from large p^2 beginning with (3.16), then the solution will be a function of two parameters α_0 and A_0. We must select A_0 to fit the condition that the physical mass of the electron is one.

Since the asymptote to A, $A_0(1/p^2)^{3\alpha_0/4\pi}$, "replaces" the bare mass in the theory, and because at low momenta this acts in the equations for small α_0 essentially like a constant, the parameter A_0 may be regarded in that domain effectively like a "mechanical" mass. In the traditional view the mechanical mass was a parameter chosen to fit the constraint that the physical mass be m. Here, A_0 replaces it. It, however, can be calculated with perturbation techniques and has an expansion of the form $A_0=1+\alpha_0a_1+\cdots$.

We now proceed to analyze the next approximation to (3.2),

$$\frac{1}{S(p)}=\gamma p+ie_0^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha S(p')\gamma^\beta-(ie_0^2)^2\int\frac{(dp')(dp'')}{(2\pi)^8}D_{\alpha\beta}(p-p')D_{\mu\nu}(p-p'')\gamma^\alpha\times S(p')\gamma^\mu S(p'+p''-p)\gamma^\beta S(p'')\gamma^\nu. \quad (3.17)$$

In this case if we use the Landau gauge and insert $S\to 1/\gamma p'$ into the above equation, a logarithmic divergence develops in the second term. This is, of course, to be expected, and we must thus regauge so that the vertex equation corresponding to (3.17) yields a finite Z_1. This is done in Appendix B and yields

$$D_{\alpha\beta}\to\left[g_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\left(1-\frac{3\alpha_0}{8\pi}\right)\right]D(k^2). \qquad (3.18)$$

If we use the gauge (3.18) in (3.17) and insert $S\to 1/\gamma p'$, then the logarithmic divergence in the second term is canceled by the logarithmic divergence in the first term generated by the α_0 term in the gauge function. It must be remembered that (3.2) is an integral equation for $S(p)$ whose kernel and whose gauge has been expanded in a power series in α_0. Since the expansion of the gauge function contributes to this expansion of the kernel, we must consistently keep all terms of the same order in the kernel. We can solve (3.17) asymptotically by inserting (3.10) and (3.11) (with $m_0=0$) in (3.17) and linearizing the resulting equations in A and B. The terms in A^2 and B^2 fall off even more rapidly. Equation (3.17) then breaks up into two linear integral equations for A and B. The equation for B, because of the choice of gauge, has an inhomogeneous term which is finite and vanishes with $D=1/k^2$. However, as before, in order to determine the dominant behavior of B for large p^2, we need to know the corrections to the high-k limit of $D(k^2)$. The equation for $A(p)$ is written in Appendix B. On dimensional grounds one can see that there is a solution of the form $A(p^2)=A_0(1/p^2)^\epsilon$. In fact, it is clear on dimensional grounds that if we continue the expansion of (3.2) to higher orders, the asymptotic form of the solution for $A(p^2)$ will be

$$A(p^2)\sim A_0(1/p^2)^{g(\alpha_0)}, \qquad (3.19)$$

where $g(\alpha_0)$ will be given as a power series in α_0. To compute $g(\alpha_0)$ correctly to order $\alpha_0{}^n$, all terms in Eq. (3.2) up to the nth order are required. It should be noted that in general the equation for A will have an nth-order kernel containing lower order vertices which are finite, as well as "irreducible" parts which contain A. The former converge, since the lower order vertices are finite by reason of our choice of gauge in lower orders. We also assume that the irreducible terms converge if A vanishes as $p^2\to\infty$. This is clear if one employs the same arguments that are used to show that only vertex renormalizations are needed in perturbation theory. Returning to the special case of the fourth order, from Eq. (3.17) we can compute $g(\alpha_0)$ correct to order $\alpha_0{}^2$. This is done in Appendix B and we obtain

$$g(\alpha_0)=\frac{3\alpha_0}{4\pi}+\frac{21}{32}\left(\frac{\alpha_0}{\pi}\right)^2. \qquad (3.20)$$

Let us summarize our results. Under the assumptions

$$(1)\quad k^2D(k^2)-1\to 0 \quad \text{as} \quad k^2\to\infty, \qquad (3.21)$$

$$(2)\quad m_0=0. \qquad (3.22)$$

We have established according to (3.2) a perturbation procedure for the calculation of the asymptotic form of the electron Green's function. The result is

$$1/S(p)=\gamma p[1+B(p^2)]+A(p^2), \qquad (3.23)$$

where $B\to 0$ in a manner determined by the corrections to the limit (3.21), while A is given by (3.19). To fourth order the exponent $g(\alpha_0)$ is given by (3.20). Naturally, we can say little about the convergence of the power series for $g(\alpha_0)$. One might remark, however, that many fewer terms arise to contribute to g in a given order, than contribute to S in renormalized perturbation theory in the same order.

Of course, everything hinges upon the fact that $D(k^2)\sim 1/k^2$, as $k^2\to\infty$. It should be emphasized that the perturbation method developed here is valid only in the asymptotic region. (3.12), (3.17), etc., will not necessarily be valid equations treated in a nonperturbative way for finite momenta. In that domain, ordinary, renormalized perturbation theory should be used.

IV. VERTEX AND WARD'S IDENTITY

From our previous discussion it is clear that (3.5) possesses finite solutions for Γ_μ even in perturbation theory. The interest of studying this equation in a nonperturbative approximation arises when we want to compute the electrodynamic contributions to $\rho(k^2)$ according to (3.8). From (3.8) it is clear that the precise nature of the asymptotic corrections to Γ_μ must be known in order to calculate these corrections to D. We will study this problem in detail in our paper on the D function. In this section we will discuss the relation to the solutions of (3.2) and (3.5). The purpose of this discussion is to indicate that a certain amount of care must be used in giving unambiguous definitions to the conditionally convergent integrals that arise in these equations to ensure their consistency.

In the first approximation the equation for S has the form (with $m_0=0$),

$$\frac{1}{S(p)}=\gamma p+ie_0{}^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha S(p')\gamma^\beta. \qquad (4.1)$$

We have shown that in the Landau gauge, the right-hand side contains no divergence. However, if we assume that

$$S(p)\sim 1/\gamma p \qquad (4.2)$$

as $p\to\infty$, then the integral

$$ie_0{}^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha\frac{1}{\gamma p'}\gamma^\beta \qquad (4.3)$$

is of course, at best only conditionally convergent. If the evaluation is made with the rule that the integral is performed using hyperspherical coordinates and the integrations over the angles of the vector p' are done first, then the value zero is obtained for (4.3). We shall assume that it is possible to evaluate such conditionally convergent integrals in each order, to obtain the value zero, so that a finite Z_2 will mean that $S(p)$ is always asymptotically $1/\gamma p$. We give such a method below. This method will be equivalent to a calculation of Z_2 from the imaginary part of the mass operator by means of the spectral representation

$$\frac{1}{S(p)}=\gamma p-\int dk\frac{r(k)}{k+\gamma p}. \quad (4.4)$$

A finite Z_2 means that $r(k)$ is such that

$$\int (dk/k^2)r(k)<\infty.$$

If we differentiate Eq. (4.1) with respect to p we find:

$$\begin{aligned}\frac{\partial}{\partial p_\lambda}\frac{1}{S(p)}&=\gamma^\lambda+ie_0^2\int\frac{(dp')}{(2\pi)^4}\frac{\partial}{\partial p_\lambda}D_{\alpha\beta}(p-p')\gamma^\alpha S(p')\gamma^\beta\\&=\gamma^\lambda-ie_0^2\int\frac{(dp')}{(2\pi)^4}\frac{\partial}{\partial p_\lambda'}(D_{\alpha\beta}(p-p')\gamma^\alpha S(p')\gamma^\beta)\\&\quad+ie_0^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha\frac{\partial}{\partial p_\lambda'}S(p')\gamma^\beta. \quad (4.5)\end{aligned}$$

If we evaluate the second term in this expression we find that it is equal to $-(3\alpha_0/8\pi)\gamma_\lambda$, so

$$\begin{aligned}\frac{\partial}{\partial p_\lambda}\frac{1}{S(p)}=\gamma_\lambda\left(1-\frac{3\alpha_0}{8\pi}\right)-ie_0^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\gamma^\alpha\\\times S(p')\frac{\partial}{\partial p_\lambda'}\left(\frac{1}{S(p')}\right)S(p')\gamma^\beta. \quad (4.6)\end{aligned}$$

Consequently, if we wish Ward's identity to hold, then the vertex equation must be (in this approximation)

$$\begin{aligned}\Gamma_\lambda(p+k,p)=\gamma_\lambda\left(1-\frac{3\alpha_0}{8\pi}\right)-ie_0^2\int\frac{(dp')}{(2\pi)^4}D_{\alpha\beta}(p-p')\\\times\gamma^\alpha S(p'+k)\Gamma_\lambda(p'+k,p')S(p')\gamma^\beta \quad (4.7)\end{aligned}$$

rather than (3.5), which was obtained from the formal derivation. We see that this minor ambiguity in momentum-space integration (or in coordinate space, in functional differentiation) results from the fact that the self-energy integrals are only conditionally convergent. Thus, free changes of the intermediate-momentum variables is not permitted. This defect does not persist in the equation for the vertex and one may freely translate the momentum variables by constant vectors. We may thus anticipate that in general the vertex equation will have an inhomogeneous term which is a finite function of α_0 multiplying γ_μ, which can be evaluated in any order by replacing, for zero momentum transfer at the vertex, $S(p)$ and $D(k)$ and $\Gamma_\mu(p,p)$ by their exact asymptotic forms $(1/\gamma p,1/k^2,\gamma_\mu)$ in the integrals. The result will be that the infinite series in the equation for Γ_μ will give a series of constants multiplying γ_μ, since Γ_μ is dimensionless, and all logarithmic divergences have been canceled. Thus, with an appropriate choice for the constant we will get $\Gamma_\mu\to\gamma_\mu$. This will guarantee that the asymptotic forms for S and Γ_μ are consistent, and that Ward's identity (gauge covariance) will be satisfied. Since in the vertex equation, free translation of the momentum variables is permitted, we shall assume as a standard form that the external momentum always appears in the electron functions and not in the photon functions. Then if we go backwards to get the equation for S from that of Γ_μ, the integration by parts in (4.5) is avoided. However, we then obtain, as our equation for $1/S(p)$, one in which the inhomogeneous term γp multiplies the constant $C(\alpha_0)$. However, we know from the discussion above that $1/S\to\gamma p$. In fact, it is easily verified that if we translate the momentum in (4.7) and integrate, we get instead of (4.1)

$$\frac{1}{S(p)}=\gamma p\left(1-\frac{3\alpha_0}{8\pi}\right)+ie_0^2\int\frac{(dk)}{(2\pi)^4}D_{\alpha\beta}(k)\gamma^\alpha S(p+k)\gamma^\beta,$$

where the rule is now to integrate symmetrically over k. If we proceed in this manner in the general case, we obtain an unambiguous equation for S with the asymptotic behavior for S, $S\to 1/\gamma p$.

V. CONCLUSIONS

We have developed a perturbation theory for quantum electrodynamics based upon a power-series expansion in the "bare" or unrenormalized coupling constant α_0, which yields a finite result for the "renormalization" constants $Z_1(=Z_2)$ and self-mass δm, provided only that $m_0=0$, that is, that the electron mass is totally dynamical. The only assumption made is that this same perturbation theory and all other interactions yield a finite value for the charge renormalization Z_3. We have found that this solution is obtained without restriction on the coupling constant.[1] We have seen that the mass term was obtainable as the result of the fact that a homogeneous linear integral equation has finite solutions for all values of the coupling parameter α_0.[7] Thus, one can view the mass as self-generating in virtue of the singular nature of the relativistic coupling which provides an integral equation

[7] This kind of idea was first clearly expressed by Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).

with a continuous eigenvalue spectrum. If we attempt to "force" a mass on the field by driving the system with a mechanical mass, then we obtain divergences. The requirement of a finite physical mass then acts as a normalization condition which fixes the scale of the solution of the homogeneous integral equation. We accordingly see that the "symmetry-breaking" character of this solution is the result of the fact that there is a certain "eigenvalue" character for the coupling which, however, holds for a continuous spectrum of eigenvalues, rather than for a discrete set as in nonrelativistic examples. As has been previously suggested, the fact that this integral equation has such solutions does not necessarily imply[8] the existence of zero-mass scalar (or pseudoscalar) states.[9]

From the point of view of the renormalized functions, the result we have obtained could be expressed by observing that if one could compute $S(p)$ to all orders in α, and then examine the asymptotic behavior of the sum, one should find that, in our gauge, $S(p)\to 1/\gamma p+(1/p^2)A$, where $A\to 0$. That is, if one computed the mechanical mass, one would obtain, as a result, zero, not, as usually is assumed, a result proportional to the physical mass.

There remains the basic problem of studying the charge renormalization. We shall do this in a subsequent paper.

We also see that it does not appear as if within a purely electrodynamic context that the μ-e mass difference can be understood.[10] The coupling between μ and e comes only through the vacuum polarization and our basic assumption (Z_3 finite) has been that this does not influence the asymptotic form of S. Hence it would appear as if an arbitrary mass ratio is allowed, since asymptotically, the electron and the μ-meson Green's function would obey uncoupled homogeneous equations, and hence we would seem to have the freedom of two constants A_{0e}, $A_{0\mu}$ to be used to fit an arbitrary mass ratio.

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APPENDIX A

When the vector p^2 is space-like (>0) we can use hyperspherical coordinates in Euclidean space. In this case (3.15) becomes

$$A(p^2)=\frac{3\alpha_0}{4\pi}\int_0^\infty dp'^2\frac{1}{p_>^2}A(p'^2)\,, \qquad \text{(A1)}$$

where we use the fact that the average of $D(p-p')$ over all angles of the four-vector p' is simply

$$\langle D(p-p')\rangle=\frac{1}{p_>^2}\begin{matrix}=1/p^2, & p>p',\\ =1/p'^2, & p'>p.\end{matrix} \qquad \text{(A2)}$$

Equation (A1) can be solved either by simply substituting $A=(1/p^2)^\epsilon A_0$, or by observing that the solution to (A1) also obeys the differential equation with $p^2=x$,

$$\frac{d}{dx}\left(x^2\frac{dA}{dx}\right)=-\frac{3\alpha_0}{4\pi}A(x)\,. \qquad \text{(A3)}$$

If we put $A(x)=(1/x^\epsilon)A_0$ we obtain from the differential equation two solutions

$$\epsilon=\tfrac{1}{2}[1\pm(1-3\alpha_0/\pi)^{1/2}]. \qquad \text{(A4)}$$

Since, however, (A2) is only valid for small α_0, this means

$$\epsilon=1-3\alpha_0/4\pi\,,$$
$$=3\alpha_0/4\pi\,.$$

Although both of these solutions obey (A1) separately, only the lower one is consistent with the requirement that the asymptotic domain (large $p'\gtrsim p$) contribute primarily to the function A.

APPENDIX B

To determine the gauge in which Z_2 is finite in this case it is most convenient to use the equation for the vertex rather than for S. To obtain the "divergent" parts we can then put $S\to 1/\gamma p$, $D\to 1/k^2$ and the external momentum equal to zero. If we let

$$D_{\alpha\beta}{}^{\alpha_0}=\left[g_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\left(1+\frac{\alpha_0}{4\pi}\gamma\right)\right]\frac{1}{k^2} \qquad \text{(B1)}$$

in the second-order term and use $D_{\alpha\beta}{}^0$ in the fourth-order term and require a cancellation of divergence, we obtain an expression for γ. Thus we require that

$$\begin{aligned}&ie_0^2\int\frac{(dk)}{(2\pi)^4}D_{\alpha\beta}{}^{\alpha_0}\gamma^\alpha\frac{1}{\gamma k}\gamma_\lambda\frac{1}{\gamma k}\gamma^\beta\\ &\quad+(ie_0^2)^2\int\frac{(dk_1)(dk_2)}{(2\pi)^8}D_{\alpha\beta}{}^0(k_1)D_{\mu\nu}{}^0(k_2)\gamma^\alpha\\ &\quad\times\left[\frac{1}{\gamma k_1}\gamma_\lambda\frac{1}{\gamma k_1}\gamma_\mu\frac{1}{\gamma k_3}\gamma_\beta\frac{1}{\gamma k_2}+\frac{1}{\gamma k_1}\gamma_\mu\frac{1}{\gamma k_3}\gamma_\lambda\frac{1}{\gamma k_3}\gamma_\beta\frac{1}{\gamma k_2}\right.\\ &\quad\left.+\frac{1}{\gamma k_1}\gamma_\mu\frac{1}{\gamma k_3}\gamma_\beta\frac{1}{\gamma k_2}\gamma_\lambda\frac{1}{\gamma k_2}\right]\gamma^\nu \qquad \text{(B2)}\end{aligned}$$

be finite, where $k_3=k_1+k_2$. Only the middle term, corresponding to the first fourth-order graph in Fig. 3, diverges. To cancel that with the α_0^2 contribution from the second-order graph requires $\gamma=-\frac{3}{2}$.

We then may compute the contributions of the fourth-order terms to the mass operator. To do this we substitute into (3.2) for the electron Green's function the expression

$$S\to 1/\gamma p+A(p^2)/p^2,$$

and in the fourth-order terms retain only the contributions linear in A. We then obtain the equation for A, accurate in the asymptotic region in hyperspherical coordinates.

$$\begin{aligned} A(p^2)={}&e_0^2\left(3+\frac{3\alpha_0}{8\pi}\right)\int\frac{(dp')}{(2\pi)^4}D(p-p')\frac{A(p'^2)}{p'^2}\\ &-e_0^4\int\frac{(dp')(dp'')}{(2\pi)^8}D_{\alpha\beta}(p-p')D_{\mu\nu}(p-p'')\\ &\times\gamma^\alpha\left[\frac{A(p'^2)}{p'^2}\gamma^\mu\frac{1}{\gamma(p'+p''-p)}\gamma^\beta\frac{1}{\gamma p''}\right.\\ &+\frac{1}{\gamma p'}\gamma^\mu\frac{A(p'+p''-p)}{(p'+p''-p)^2}\gamma^\beta\frac{1}{\gamma p''}\\ &\left.+\frac{1}{\gamma p'}\gamma^\mu\frac{1}{\gamma(p'+p''-p)}\gamma^\beta\frac{A(p''^2)}{p''^2}\right]\gamma^\nu. \end{aligned}\tag{B3}$$

It follows immediately by counting powers that if $A\to A_0(1/p^2)^\epsilon$ then the second term $\to A_0(1/p^2)^\epsilon f(\epsilon)$ providing only that $\epsilon>0$ (that is, that the integrals converge). To evaluate for small ϵ, one may use the representation

$$\begin{aligned}\left(\frac{1}{p^2}\right)^{1+\epsilon}&=\int_0^\infty\frac{d\lambda}{\lambda^\epsilon}\frac{1}{(\lambda+p^2)^2}n(\epsilon),\\ 1&=n(\epsilon)\int_0^\infty\frac{d\lambda}{\lambda^\epsilon}\frac{1}{(\lambda+1)^2},\end{aligned}\tag{B4}$$

where for small ϵ, $n(\epsilon)\to 1$. In this case, if one carries out integrations over the momenta, one will be left with a function of p^2 and λ, which as $\lambda\to\infty$ will approach λ_0/λ. That is, the second term will be proportional to

$$\int_0^\infty\frac{d\lambda}{\lambda^\epsilon}g(\lambda,p^2),$$

where for $\lambda\gg p^2$, $g(\lambda,p^2)\to\lambda_0/\lambda$. But this means that if we want only the contributions of the second term of order $1/\epsilon$, these will be

$$\sim\int_{\gg p^2}^\infty\frac{d\lambda}{\lambda^\epsilon}\frac{\lambda_0}{\lambda}\sim\int_{\gg 1}^\infty\frac{dz}{z^\epsilon}\frac{\lambda_0}{z}\left(\frac{1}{p^2}\right)^\epsilon\sim\frac{\lambda_0}{\epsilon}\left(\frac{1}{p^2}\right)^\epsilon.\tag{B5}$$

Therefore, we need only compute λ_0, which is defined by the equation

$$\begin{aligned}\lambda_0={}&-e_0^4\lambda\int\frac{(dp')(dp'')}{(2\pi)^8}D_{\alpha\beta}(p')D_{\mu\nu}(p'')\\ &\times\gamma^\mu\left[\gamma^\mu\frac{1}{\gamma(p'+p'')}\gamma^\beta\frac{1}{\gamma p''}\frac{1}{(p'^2+\lambda)^2}\right.\\ &+\frac{1}{\gamma p'}\gamma^\mu\gamma^\beta\frac{1}{\gamma p''}\frac{1}{[(p'+p'')^2+\lambda]^2}\\ &\left.+\frac{1}{\gamma p'}\gamma^\mu\frac{1}{\gamma(p'+p'')}\gamma^\beta\frac{1}{(p''^2+\lambda)^2}\right]\gamma^\nu.\end{aligned}\tag{B6}$$

If we evaluate this we find

$$\lambda_0=0$$

(that is, the graph is finite in perturbation theory in this gauge). Therefore, we obtain the equation for ϵ correct to the second order in α_0,

$$1=\frac{3\alpha_0}{4\pi}\left(1+\frac{\alpha_0}{8\pi}\right)\frac{1}{\epsilon(1-\epsilon)}$$

so

$$\epsilon=\frac{3\alpha_0}{4\pi}+\frac{21}{32}\left(\frac{\alpha_0}{\pi}\right)^2.\tag{B7}$$